JOURNAL OF APPROXIMATION THEORY **87**, 338–359 (1996)
ARTICLE NO. 0108

Riesz Bases of Splines and Regularized Splines with Multiple Knots

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Communicated by C. K. Chui

Received April 23, 1995; accepted January 2, 1996

This paper deals with $L_2(\mathbf{R})$-norm and Sobolev-norm stability of polynomial splines with multiple knots, and with regularized versions thereof. An essential ingredient is a result on Hölder continuity of the shift operator operating on a B-spline series. The stability estimates can be reformulated in terms of a Riesz basis property for the underlying spline spaces. These can also be employed to derive a result on stable Hermite interpolation on the real line. We point to the connection with the problem of symmetric preconditioning of bi-infinite interpolation matrices.

1. NOTATIONS AND OUTLINE OF THE PAPER

The motivation for our present work stems from the well-known stability result for B-spline series due to de Boor [2]. In [10] we have developed a generalization for regularized splines with simple knots as defined by certain tempered distributions. These stability estimates in $L_2(\mathbf{R})$ are of particular importance for orthogonal decomposition of spaces which are spanned by arbitrary shifts of one generating function, see [5]. Such function spaces occur in a host of applications of radial basis functions to interpolation and approximation of scattered data (Powell [13]).

The purpose of this writing is threefold. First, in Section 2 we give an extension of the L_2-stability estimate to the case where Sobolev norms are involved. The main result (see Theorem 2.3) applies to splines with multiple knots and is based on the Hölder continuity of the shift operator operating on a B-spline series (Lemma 2.1). Second, Section 3 deals with stability estimates for regularized splines based on tempered distributions satisfying Assumption 1 below; the main result in this section, Theorem 3.1, extends

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0021-9045/96 $18.00

our stability estimate in [10, Theorem 1] to the case of multiple knots. Theorem 3.2 gives a reformulation of the stability result in terms of a Riesz basis property for the corresponding space of regularized spline functions. Third, in Section 4 we show that scattered Hermite interpolation on the real line is regular subject to condition (3.1) being satisfied. This generalizes a well known result on even order scattered Lagrange interpolation on the real line, but at the same time adds the new result on stability for this operation.

Let us begin with the notations involved. An (*extended*) *knot sequence of order* $m \in \mathbf{N}$ is given by the ordered set

$$X = \{ \cdots \leqslant x_i \leqslant x_{i+1} \cdots \}, \quad x_i < x_{i+m}, \tag{1.1}$$

of real numbers, subject to $\lim_{i \to -\infty} x_i = -\infty$ and $\lim_{i \to +\infty} x_i = +\infty$, and the *multiplicity* (*or order*) *of a knot* $x_i \in X$ is defined as $\operatorname{ord}(x_i) := \# \{ j \in \mathbf{Z};\, x_j = x_i \}$.

The *divided differences of order* $k \geqslant 0$ with respect to the knot sequence X are recursively given by

$$\nu_{i,k}(f) := \begin{cases} \dfrac{1}{k!} f^{(k)}(x_i), & \text{if} \quad x_i = x_{i+k}, \\ \dfrac{\nu_{i,k-1}(f) - \nu_{i+1,k-1}(f)}{x_i - x_{i+k}}, & \text{if} \quad x_i < x_{i+k}. \end{cases} \tag{1.2}$$

They are used to define the sequence of B-*splines of order m associated with* X in the usual way, viz.

$$B_{i,m}(t) := m\, \nu_{i,m}((\cdot - t)_+^{m-1}), \qquad i \in \mathbf{Z}\,. \tag{1.3}$$

(Here and in what follows, identities must be interpreted in the weak sense whenever a knot of maximal order m is involved.) Alternately, based on the fundamental solution $h(x) := x_+^{m-1}/(m-1)!$ of the differential operator $D^m\colon f \mapsto f^{(m)}$ we have the representation

$$B_{i,m}(t) = m\,(-1)^m\, \nu_{i,m}((t - \cdot)_+^{m-1}) = m!\,(-1)^m\,(h * \nu_{i,m})(t), \quad i \in \mathbf{Z}, \tag{1.4}$$

which will be more appropriate to our purposes. (The convolution is in terms of convolution of tempered distributions.) These B-splines are piecewise polynomial functions of local degree less than m, and with breakpoints at the knots X. They are normalized so as to give $\int_{-\infty}^{+\infty} B_{i,m}(t)\, dt = 1$, and their support is $\operatorname{supp} B_{i,m} = [x_i, x_{i+m}]$. They can also be considered as the

Peano kernels for the divided differences of order m, since we have by Peano's Theorem that

$$v_{i,m}(f) = \frac{1}{m!} \int_{-\infty}^{+\infty} B_{i,m}(t)\, f^{(m)}(t)\, dt \tag{1.5}$$

for any $f \in C^{m-1}(\operatorname{supp} B_{i,m})$ with absolutely continuous $(m-1)$st derivative.

The *Fourier transform of the* B-*splines* can be derived as follows. We have ([7, p. 360])

$$h^\wedge(\xi) = (i\xi)^{-m} + \frac{i^{m-1}}{(m-1)!}\, \pi \delta^{(m-1)}.$$

In addition, $v_{i,m}$ having compact support, its Fourier transform is an entire function which, since $v_{i,m}$ annihilates polynomials of degree less than m, has an mth order zero at the origin,

$$v^\wedge_{i,m}(\xi) = \mathcal{O}(\xi^m) \qquad \text{as} \quad \xi \to 0.$$

From this, Eq. (1.4) yields

$$B^\wedge_{i,m}(\xi) = m!\,(-1)^m\, h^\wedge(\xi)\, v^\wedge_{i,m}(\xi) = m!\; v^\wedge_{i,m}(\xi)/(-i\xi)^m, \tag{1.4a}$$

which is an entire function of exponential type. For fixed m, it depends continuously on the knots $x_i, ..., x_{i+m}$ (in the sense of compact convergence).

We also need two results, well known from the spline literature, where we put

$$d_{j,k} := \frac{x_{j+k} - x_j}{k}, \qquad k \in \mathbf{N}, \quad j \in \mathbf{Z}. \tag{1.6}$$

The first one deals with the L_2-stability of the B-spline series, due to de Boor [2].

LEMMA 1.1 (See [6, Chap. 5, Theorem 4.2]). *For the knot sequence* (1.1) *there exists a constant* $D_m > 0$ *depending only on m such that*

$$D_m \|v\|_{\ell_2(\mathbf{Z})} \leqslant \Big\| \sum_{j \in \mathbf{Z}} v_j \sqrt{d_{j,m}}\, B_{j,m} \Big\|_{L_2(\mathbf{R})} \leqslant \|v\|_{\ell_2(\mathbf{Z})} \tag{1.7}$$

for any sequence $v = (v_j)_{j \in \mathbf{Z}}$.

In this lemma, we need not assume that $v \in \ell_2(\mathbf{Z})$ if we allow $\|v\|_{\ell_2(\mathbf{Z})}$ to take the value ∞, and if we interpret inequalities correspondingly. However, in case $v \in \ell_2(\mathbf{Z})$ we see that the map

$$\sigma \colon v \mapsto \sigma_v := \sigma_{v,m} := \sum_{j \in \mathbf{Z}} v_j \sqrt{d_{j,m}}\, B_{j,m} \tag{1.8}$$

(which is locally defined for any sequence v) represents a homeomorphism between $\ell_2(\mathbf{Z})$ and

$$S_{m,X} := \operatorname{clos}_{L_2(\mathbf{R})} S^0_{m,X} \qquad \text{with} \quad S^0_{m,X} := \operatorname{span}\{B_{j,m};\, j \in \mathbf{Z}\}, \tag{1.9}$$

the latter considered as subspaces of $L_2(\mathbf{R})$; in other words, $\{\sqrt{d_{j,m}}\, B_{j,m};$ $j \in \mathbf{Z}\}$ *is a Riesz basis of the closed subspace* $S_{m,X}$.

The second result is the well-known differentiation formula for a B-spline series (see [6, Chap. 5, Eq. (3.11)]) adapted to our notation.

LEMMA 1.2. *Let $m > 1$ and $d_{j,m-1} > 0$, $j \in \mathbf{Z}$. Then for any sequence v and the corresponding spline $\sigma_{v,m}$ as in* (1.7), *we have*

$$\sigma'_{v,m} = \sum_{j \in \mathbf{Z}} w_j \sqrt{d_{j,m-1}}\, B_{j,m-1}$$

with

$$w_j = d_{j,m-1}^{-1/2} (d_{j,m}^{-1/2} v_j - d_{j-1,m}^{-1/2} v_{j-1}), \qquad j \in \mathbf{Z};$$

i.e., we have

$$\sigma'_{v,m} = \sigma_{w,m-1} \qquad \text{with} \quad w := D_m v,$$

where $D_m = (d^m_{j,k})_{j,k \in \mathbf{Z}}$ is the lower two-banded matrix with entries

$$d^m_{j,j} := (d_{j,m-1} d_{j,m})^{-1/2}, \qquad d^m_{j,j-1} := -(d_{j,m-1} d_{j-1,m})^{-1/2}, \qquad j \in \mathbf{Z},$$

and $d^m_{j,k} = 0$ otherwise.

This differentiation formula can also be expressed in terms of the modified divided differences

$$\mu_{i,k} := k!\, \sqrt{d_{i,k}}\, v_{i,k}\,, \quad k \in \mathbf{N}\,, \quad i \in \mathbf{Z}\,, \tag{1.10}$$

viz. with the assumptions and notations of Lemma 1.2 we have

$$\mu_{i,m} = d_{i,m}^{-1/2} (d_{i+1,m-1}^{-1/2} \mu_{i+1,m-1} - d_{i,m-1}^{-1/2} \mu_{i,m-1}),$$

i.e.,

$$\mu^{(m)} := (\mu_{i,\,m})_{i \in \mathbf{Z}} = -D_m^* \mu^{(m-1)} \qquad \text{if} \quad d_{i,\,m-1} > 0, \quad i \in \mathbf{Z}. \tag{1.11}$$

2. SOBOLEV NORM STABILITY OF SPLINES

For the values $d_{j,\,k}$ as defined in (1.6), we put

$$q_k := \inf_{j \in \mathbf{Z}} d_{j,\,k}, \qquad k \in \mathbf{N}. \tag{2.1}$$

In this section we derive a stability estimate for B-spline series in terms of Sobolev norms

$$\|f\|_{H^s(\mathbf{R})} := \left(\frac{1}{2\pi} \int_{\mathbf{R}} (1 + |\xi|^2)^s \, |f^\wedge(\xi)|^2 \, d\xi \right)^{1/2}, \tag{2.2}$$

with $0 \leqslant s \in \mathbf{R}$ arbitrary. It is clear that $L_2(\mathbf{R}) = H^0(\mathbf{R})$, and $\|f\|_{L_2(\mathbf{R})} \leqslant \|f\|_{H^s(\mathbf{R})}$. In particular, this shows that the left-hand estimate of (1.6) carries over to Sobolev spaces,

$$D_m \, \|v\|_{\ell_2(\mathbf{Z})} \leqslant \left\| \sum_{j \in \mathbf{Z}} v_j \sqrt{d_{j,\,m}} \, B_{j,\,m} \right\|_{H^s(\mathbf{R})} \qquad \text{for any} \quad 0 \leqslant s \in \mathbf{R}. \tag{2.3}$$

Here again, either of the expressions may take the value infinity with corresponding interpretation of the inequality.

Estimating the Sobolev norm from above is more involved. It is helpful to use the following equivalent norm in $H^s(\mathbf{R})$ (see [8, Chap. 7.9]): Let $s = k + t$ with k the integer part of s and $0 \leqslant t < 1$; then for some constants $0 < c_1 \leqslant c_2 < \infty$ depending only on k,

$$\begin{aligned} c_1 \, \|f\|^2_{H^s(\mathbf{R})} &\leqslant \sum_{\kappa=0}^{k} \|f^{(\kappa)}\|^2_{L_2(\mathbf{R})} + A_t \iint_{\mathbf{R}^2} \frac{|f^{(k)}(x) - f^{(k)}(y)|^2}{|x-y|^{1+2t}} \, dx \, dy \\ &\leqslant c_2 \, \|f\|^2_{H^s(\mathbf{R})} \end{aligned} \tag{2.4}$$

with $A_0 := 0$ and $A_t^{-1} := \int_{\mathbf{R}} |e^{ix} - 1|^2 / |x|^{1+2t} \, dx$ for $0 < t < 1$. We note that $A_t / t(1-t)$ is bounded as $t \to 0$ and $t \to 1$ [8, p. 241].

LEMMA 2.1. *Let $q_m > 0$. Then, for any $v \in \ell_2(\mathbf{Z})$ and the corresponding spline*

$$f := \sigma_v := \sum_{j \in \mathbf{Z}} v_j \sqrt{d_{j,\,m}} \, B_{j,\,m},$$

we have

$$\|f(\cdot+x)-f\|^2_{L_2(\mathbf{R})} \leqslant \text{const}\,\frac{|x|}{q_m}\,\|v\|^2_{\ell_2(\mathbf{Z})} \qquad \textit{for} \quad |x|<q_m, \tag{2.5}$$

where the constant depends only on m.

Proof. It is sufficient to consider the case $x \geqslant 0$. We write $q := q_m$ for short. By reducing the order of all knots of X to order one, we arrive at the set

$$X = \Xi = \{\,\cdots < \xi_j < \xi_{j+1} < \cdots\}.$$

Let $M_x := \bigcup_{j\in\mathbf{Z}}[\xi_j - x,\ \xi_j]$, and let us estimate the terms in

$$\begin{aligned}\|f(\cdot+x)-f\|^2_{L_2(\mathbf{R})^2}\\ &= \int_{M_x} |f(y+x)-f(y)|^2\,dy + \int_{\mathbf{R}\setminus M_x} |f(y+x)-f(y)|^2\,dy\end{aligned}$$

separately.

First let $y \in I_j(x) := [\xi_j - x,\ \xi_j]$. Then $y,\ y+x \in [\xi_j - x,\ \xi_j + x]$, and we put

$$K_j := \{k \in \mathbf{Z}\,;\, [\xi_j - x,\ \xi_j + x] \cap \operatorname{supp} B_{k,m} \neq \varnothing\}.$$

From this

$$\begin{aligned}|f(y+x)-f(y)|^2 &\leqslant \Big(\sum_{k\in K_j} |v_k|\,\sqrt{d_{k,m}}\,|B_{k,m}(y+x)-B_{k,m}(y)|\Big)^2\\ &\leqslant \sum_{k\in K_j} |v_k|^2 \sum_{k\in\mathbf{Z}} d_{k,m}\,|B_{k,m}(y+x)-B_{k,m}(y)|^2\\ &\leqslant 2\sum_{k\in K_j} |v_k|^2 \Big\|\sum_{k\in\mathbf{Z}} d_{k,m}\,B^2_{k,m}\Big\|_{L_\infty(\mathbf{R})}\\ &\leqslant \frac{2}{q}\sum_{k\in K_j} |v_k|^2,\end{aligned}$$

the latter inequality following from the fact that $\sum_{k\in\mathbf{Z}} d_{k,m}\,B_{k,m} \equiv 1$ and hence $0 \leqslant B_{k,m} \leqslant d^{-1}_{k,m} \leqslant q^{-1}$. Integrating over M_x yields

$$\begin{aligned}\int_{M_x} |f(y+x)-f(y)|^2\,dy &\leqslant \sum_{j\in\mathbf{Z}} \int_{I_j(x)} |f(y+x)-f(y)|^2\,dy\\ &\leqslant \frac{2x}{q}\sum_{j\in\mathbf{Z}}\sum_{k\in K_j} |v_k|^2.\end{aligned}$$

Now $\xi_j = x_{\ell_j}$ for some $\ell_j \in \mathbf{Z}$ with $\ell_j < \ell_{j+1}$, and since $0 \leqslant x < q$ we have $x_{\ell_j - m} < \xi_j - x \leqslant \xi_j + x < x_{\ell_j + m}$. From this we get the final estimate

$$\begin{aligned}\int_{M_x} |f(y+x) - f(y)|^2 \, dy &\leqslant \frac{2x}{q} \sum_{j \in \mathbf{Z}} \sum_{k = \ell_j - 2m + 1}^{\ell_j + m - 1} |v_k|^2 \\ &\leqslant \frac{2x}{q} (3m-1) \|v\|^2_{\ell_2(\mathbf{Z})}. \end{aligned} \tag{2.6}$$

Now let us consider the set

$$\mathbf{R} \backslash M_x = \textstyle\bigcup_{j \in J_x} (\xi_j, \xi_{j+1} - x) \qquad \text{with} \quad J_x := \{ j \in \mathbf{Z} : \ \xi_{j+1} - \xi_j > x \}.$$

Since f is a polynomial of degree less than m on each component of this set, we have for $j \in J_x$ that

$$\begin{aligned} &\int_{\xi_j}^{\xi_{j+1} - x} |f(y+x) - f(y)|^2 \, dy \\ &\quad = \int_{\xi_j}^{\xi_{j+1} - x} \left| \int_y^{y+x} f'(z) \, dz \right|^2 dy \\ &\quad \leqslant x \int_{\xi_j}^{\xi_{j+1} - x} \int_y^{y+x} |f'(z)|^2 \, dz \, dy \leqslant x \int_{\xi_j}^{\xi_{j+1}} |f'(z)|^2 \int_{z-x}^{z} dy \, dz \\ &\quad \leqslant x^2 \int_{\xi_j}^{\xi_{j+1}} |f'(z)|^2 \, dz \leqslant c_m \frac{x^2}{\xi_{j+1} - \xi_j} \|f\|^2_{L_\infty([\xi_j, \xi_{j+1}])} \end{aligned}$$

the latter according to an L_2-version of Markov's inequality (see [6, Chap. 4, Theorem 2.7]); here, the constant c_m depends only on m. Now for $\xi_j < y < \xi_{j+1}$ and $\xi_j = x_{\ell_j} < x_{\ell_j + 1} = \xi_{j+1}$,

$$\begin{aligned} \left| \sum_{k \in \mathbf{Z}} v_k \sqrt{d_{k,m}} \, B_{k,m}(y) \right|^2 &= \left| \sum_{k = \ell_j - m + 1}^{\ell_j} v_k \sqrt{d_{k,m}} \, B_{k,m}(y) \right|^2 \\ &\leqslant \sum_{k = \ell_j - m + 1}^{\ell_j} |v_k|^2 \sum_{k = \ell_j - m + 1}^{\ell_j} d_{k,m} \, B^2_{k,m}(y) \\ &\leqslant \frac{1}{q} \sum_{k = \ell_j - m + 1}^{\ell_j} |v_k|^2, \end{aligned}$$

using the same estimate as above. Since $\xi_{j+1} - \xi_j > x$,

$$\int_{\xi_j}^{\xi_{j+1} - x} |f(y+x) - f(y)|^2 \, dy \leqslant c_m \frac{x}{q} \sum_{k = \ell_j - m + 1}^{\ell_j} |v_k|^2.$$

Summing over all $j \in J_x$ yields

$$\int_{\mathbf{R}\setminus M_x} |f(y+x)-f(y)|^2\,dy \leqslant c_m \frac{mx}{q}\,\|v\|^2_{\ell_2(\mathbf{Z})}. \tag{2.7}$$

Combining this with (2.6) proves the lemma. We also see that the constant in (2.5) can be written as $m(6+c_m)-2$. ∎

Remark. Combining this estimate with Lemma 1.1 shows that, at every $f \in S_{m,X}$, the translation operator $f \mapsto f(\cdot+x)$ is Hölder continuous with Hölder exponent $\frac{1}{2}$. This result could be derived in a different way for splines with simple knots (using the arguments of the proof of [10, Theorem 1.b]). It is interesting that this Hölder continuity has been derived without any restriction on the multiplicity of the knots, with the sole assumption that the width of the support of the B-splines is uniformly bounded away from zero.

LEMMA 2.2. *Let $q_m>0$ and $0<t<\frac{1}{2}$. Then, for any $v \in \ell_2(\mathbf{Z})$ and the corresponding spline $f := \sigma_v := \sum_{j\in\mathbf{Z}} v_j \sqrt{d_{j,m}}\,B_{j,m}$, we have*

$$\iint_{\mathbf{R}^2} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2t}}\,dx\,dy \leqslant \text{const}\,\frac{1}{t(1-2t)}\,\|v\|^2_{\ell_2(\mathbf{Z})}, \tag{2.8}$$

where the constant depends only on m and q_m.

Proof. This follows immediately from Lemma 1.1 and Lemma 2.1, since, with $q=q_m$, the integral can be estimated as

$$\begin{aligned}
&\iint_{\mathbf{R}^2} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2t}}\,dx\,dy\\
&= \int_{|x|\geqslant q} \frac{1}{|x|^{1+2t}}\left(\int_{\mathbf{R}} |f(x+y)-f(y)|^2\,dy\right)dx\\
&\quad+\int_{|x|<q} \frac{1}{|x|^{1+2t}}\left(\int_{\mathbf{R}} |f(x+y)-f(y)|^2\,dy\right)dx\\
&\leqslant 4\,\|f\|^2_{L_2(\mathbf{R})}\int_{|x|\geqslant q}\frac{1}{|x|^{1+2t}}\,dx+\text{const}\,\frac{1}{q}\,\|v\|^2_{\ell_2(\mathbf{Z})}\int_{|x|<q}\frac{1}{|x|^{2t}}\,dx\\
&\leqslant\left(\frac{4}{tq^{2t}}+\text{const}\,\frac{2}{q^{2t}(1-2t)}\right)\|v\|^2_{\ell_2(\mathbf{Z})},
\end{aligned}$$

with the same constant as in Lemma 2.1. ∎

We are now ready to prove the main result of this section.

THEOREM 2.3. *Let* $\mu \in \{1, ..., m\}$ *with* $q_\mu > 0$, *and let* $0 \leqslant t < \frac{1}{2}$. *Then, for any* $v \in \ell_2(\mathbf{Z})$ *and the corresponding spline* $f := \sigma_{v,m} := \sum_{j \in \mathbf{Z}} v_j \sqrt{d_{j,m}}\, B_{j,m}$, *we have the stability estimates*

$$D_m \|v\|_{\ell_2(\mathbf{Z})} \leqslant \Big\| \sum_{j \in \mathbf{Z}} v_j \sqrt{d_{j,m}}\, B_{j,m} \Big\|_{H^{m-\mu+t}(\mathbf{R})} \leqslant \text{const}\, \|v\|_{\ell_2(\mathbf{Z})}, \tag{2.9}$$

where the constant can be chosen only depending on m, t, *and* q_μ. *Moreover, if* $0 \leqslant t \leqslant t_0 < \frac{1}{2}$, *then the constant can be chosen depending on* m, t_0, *and* q_μ.

Proof. We only have to verify the upper estimate. Here, we employ (2.4) with $k = m - \mu$. Since $q_\ell > 0$ for $\ell = \mu, ..., m$, we can apply Lemma 1.2 in order to see that

$$\sigma^{(\kappa)}_{v,m} = \sigma_{w^{(\kappa)},\, m-\kappa} \qquad \text{with} \quad w^{(\kappa)} := D_{m-\kappa+1} D_{m-\kappa+2} \cdots D_m v, \quad \kappa = 1, ..., k.$$

The entries of the lower two-banded matrices involved in this representation are uniformly bounded, and hence

$$\|w^{(\kappa)}\|^2_{\ell_2(\mathbf{Z})} \leqslant c_{q_\mu} \|v\|^2_{\ell_2(\mathbf{Z})}, \qquad \kappa = 0, ..., k,$$

with the constant depending only on q_μ. By Lemma 1.1, this shows that

$$\sum_{\kappa=0}^{m-\mu} \|\sigma^{(\kappa)}_{v,m}\|^2_{L_2(\mathbf{R})} \leqslant (m - \mu + 1)\, c_{q_\mu} \|v\|^2_{\ell_2(\mathbf{Z})}. \tag{2.10}$$

In case $t > 0$, the remaining part in (2.4) is estimated by applying Lemma 2.2 to $g := \sigma_{w^{(m-\mu)},\, \mu}$ showing that

$$A_t \iint_{\mathbf{R}^2} \frac{|g(x) - g(y)|^2}{|x-y|^{1+2t}}\, dx\, dy \leqslant \text{const}\, \frac{c_{q_\mu}}{1-2t} \frac{A_t}{t} \|v\|^2_{\ell_2(\mathbf{Z})}, \tag{2.11}$$

with the same constant as in the lemma (depending on $m - \mu$ and q_μ). Combining these two estimates with the fact that A_t / t is bounded as $t \to 0$, we arrive at the required estimate. The theorem is proved. ▮

Remark. The constant in (2.9) goes to infinity as $t \to \frac{1}{2}$. This is consistent with the observation that $B_{j,m}$ fails to be in $H^{m-\mu+1/2}$ if it has a knot of order μ (whence $q_{\mu-1} = 0$).

3. RIESZ BASES OF REGULARIZED SPLINES

Throughout the rest of the paper we will often refer to the fact that X is *minimally separated of order m* by which we mean that

$$q_m := \inf_{j \in \mathbf{Z}} d_{j,m} > 0. \tag{3.1}$$

It has been observed by several authors (e.g. [4, 12]) that appropriate differencing of certain functions of polynomial growth leads to bell-shaped functions. In this section we are after such generalizations of B-splines which can be derived from the right-hand expression in (1.4) by allowing h to be a certain tempered distribution. Unlike earlier results in the literature, here we allow multiple knots, to which we can extend our stability result of [10]. As in [5] we assume the following.

Assumption 1. h is a tempered distribution, and its distributional Fourier transform, $h^\wedge$, satisfies for some $m \in \mathbf{N}$:

(i) $h^\wedge \in C(\mathbf{R} \setminus \{0\})$,

(ii) $h^\wedge(\xi) \neq 0$ for $\xi \in \mathbf{R} \setminus \{0\}$,

(iii) $(i\xi)^m h^\wedge \in L_\infty(\mathbf{R})$,

(iv) $1/((i\xi)^m h^\wedge) \in L_\infty(U_0)$ for some neighbourhood U_0 of 0.

In this case we put

$$G(\xi) := |\xi|^{2m} |h^\wedge(\xi)|^2, \tag{3.2}$$

and we define the matrix $B = (b_{i,j})_{i,j \in \mathbf{Z}}$, based on the B-splines of order m, by

$$b_{i,j} := \frac{\sqrt{d_{i,m} d_{j,m}}}{2\pi} \int_{-\infty}^{+\infty} G(\xi)\, B_{i,m}^\wedge(\xi)\, \overline{B_{j,m}^\wedge(\xi)}\, d\xi. \tag{3.3}$$

Clearly, B is hermitian, and it defines a quadratic form on $\ell_2(\mathbf{Z})$ by putting $\langle By, y\rangle := y^H By$ for any $y \in \ell_2^0$, the sequences of finite support. This definition extends to all of $\ell_2(\mathbf{Z})$, since by Assumption 1(iii) and the upper estimate in Lemma 1.1,

$$|\langle By, y\rangle| = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(\xi)\, |\sigma_{y,m}^\wedge(\xi)|^2\, d\xi \leqslant \Gamma\, \|\sigma_{y,m}\|_{L_2(\mathbf{R})}^2 \leqslant \Gamma\, \|y\|_{\ell_2(\mathbf{Z})}^2 \tag{3.4}$$

where $\Gamma := \|G\|_{L_\infty(\mathbf{R})} < \infty$. It is less obvious that the quadratic form is bounded from below as well.

THEOREM 3.1. *Let $q_m > 0$. Then there is a constant $\gamma > 0$ such that*

$$\langle By, y \rangle \geqslant \gamma \, \|y\|^2_{\ell_2(\mathbf{Z})} \tag{3.5}$$

for any $y \in \ell_2(\mathbf{Z})$.

Proof. This is a corollary of Theorem 2.3. For $f := \sigma := \sigma_{y,\,m}$ we have

$$\langle By, y \rangle = \frac{1}{2\pi} \int_{\mathbf{R}} G(\xi) \, |\sigma^\wedge(\xi)|^2 \, d\xi,$$

and by Assumption 1, for any $n > 0$ we find

$$\gamma_n := \operatorname*{ess\,inf}_{|\xi| \leqslant n} G(\xi) > 0.$$

This yields the estimate

$$\begin{aligned} \langle By, y \rangle &\geqslant \gamma_n \left\{ \frac{1}{2\pi} \int_{\mathbf{R}} |\sigma^\wedge(\xi)|^2 \, d\xi - \frac{1}{2\pi} \int_{|\xi| > n} |\sigma^\wedge(\xi)|^2 \, d\xi \right\} \\ &\geqslant \gamma_n \left\{ \|\sigma\|^2_{L_2(\mathbf{R})} - \frac{1}{(1+n^2)^s} \, \|\sigma\|^2_{H^s(\mathbf{R})} \right\} \end{aligned}$$

for any $s > 0$. Using the lower estimate in Lemma 1.1, and Theorem 2.3 with $\mu = m$ and $0 < t < \frac{1}{2}$, we see that

$$\langle By, y \rangle \geqslant \gamma_n \left\{ D_m^2 - \frac{\text{const}}{(1+n^2)^t} \right\} \|y\|^2_{\ell_2(\mathbf{Z})},$$

where the constant depends only on m, t, and q_m. By choosing n appropriately, we arrive at the statement of the theorem. ∎

Remark. This result generalizes our stability result in [10, Theorem 1] to the case of multiple knots. It should be noted that our earlier result implies Theorem 3.1 in case $m > 1$ and $q_{m-1} > 0$. However, our arguments used there (which are based on the differentiation formula for a B-spline series) break down in the case mth order knots are considered.

Estimates (3.4), (3.5) can be reformulated as a Riesz basis property of certain generalized splines. With the divided differences (1.2) and their modified versions taken as in (1.10), we put

$$\varphi_j := h * \mu_j \qquad \text{with} \quad \mu_j := \mu_{j,\,m} = m! \sqrt{d_{j,\,m}} \, \nu_{j,\,m}, \quad j \in \mathbf{Z}. \tag{3.6}$$

These convolution products are well-defined, since h is tempered and the support of μ_j is finite; their Fourier transforms, according to (1.4.a), are given by

$$\varphi_j^\wedge(\xi) = h^\wedge(\xi)\,\mu_j^\wedge(\xi) = (-i\xi)^m\, h^\wedge(\xi)\,\sqrt{d_{j,m}}\,B_{j,m}^\wedge(\xi), \quad j \in \mathbf{Z}, \tag{3.7}$$

and, by Assumption 1, this shows that $\varphi_j \in L_2(\mathbf{R})$ for any $j \in \mathbf{Z}$. An application of Parseval's identity thus yields the following.

THEOREM 3.2. *Suppose that* (3.1) *holds, and that Assumption* 1 *is satisfied. Then we have*

$$b_{ij} = \langle \varphi_i, \varphi_j \rangle_{L_2(\mathbf{R})} \qquad \textit{for} \quad i, j \in \mathbf{Z}; \tag{3.8}$$

i.e., B is the Gramian of the functions defined by (3.6). *In particular, by putting*

$$V_X^0 := \operatorname{span}\{\varphi_i;\, i \in \mathbf{Z}\} \qquad \textit{and} \qquad V_X := \operatorname{clos}_{L_2(\mathbf{R})} V_X^0, \tag{3.9}$$

we have that the system $\{\varphi_i;\, i \in \mathbf{Z}\}$ is a Riesz basis for V_X,

$$\gamma\,\|a\|_{\ell_2(\mathbf{Z})}^2 \leqslant \Big\| \sum_{j \in \mathbf{Z}} a_j \varphi_j \Big\|_{L_2(\mathbf{R})}^2 \leqslant \Gamma \|a\|_{\ell_2(\mathbf{Z})}^2, \tag{3.10}$$

with γ as in Theorem 3.1 *and $\Gamma = \|G\|_{L_\infty(\mathbf{R})}$.*

Due to formula (3.7), we see that the functions φ_j are closely related to the B-splines $B_{j,m}$, and hence the function space V_X is closely related to the spline space $S_{m,X}$. In case $h(x) = x_+^{m-1}/(m-1)!$ we have $(i\xi)^m\, h^\wedge(\xi) \equiv 1$; i.e., we have $\varphi_j = (-1)^m \sqrt{d_{j,m}}\, B_{j,m}$, for $j \in \mathbf{Z}$, and $V_X = S_{m,X}$. This is consistent with formula (1.4). In other cases, in particular if $h^\wedge$ has good decay properties at ∞, we have regularized versions of B-splines. For further examples and more details we refer the reader to [5].

We can also give a Sobolev norm estimate. Let

$$f = \sum_{j \in \mathbf{Z}} a_j \varphi_j \in V_X.$$

Then for any $s \geqslant 0$,

$$\|f\|_{H^s(\mathbf{R})}^2 = \frac{1}{2\pi} \int_{\mathbf{R}} (1 + |\xi|^2)^s\, G(\xi)\, |\sigma_{a,m}^\wedge(\xi)|^2\, d\xi,$$

and Theorem 2.3 combined with (3.10) implies the following.

THEOREM 3.3. *Let $\mu \in \{1, ..., m\}$ with $q_\mu > 0$, and let $0 \leqslant t < \frac{1}{2}$. Furthermore, suppose that Assumption 1 is satisfied. Then for any $s \geqslant 0$,*

$$\sqrt{\gamma}\, \|a\|_{\ell_2(\mathbf{Z})} \leqslant \left\| \sum_{j \in \mathbf{Z}} a_j \varphi_j \right\|_{H^s(\mathbf{R})} \leqslant \text{const}\, \Gamma_s\, \|a\|_{\ell_2(\mathbf{Z})}, \tag{3.11}$$

with the same constant as in Theorem 2.3, $\Gamma_s := \|(1+|\cdot|^2)^{s'} G\|_{L_\infty(\mathbf{R})}^{1/2}$ and $s' := \max\{0, s-(m-\mu+t)\}$.

Here, we again allow Γ_s to take the value ∞, but we point to the fact that, depending on the decay properties of $h^\wedge$, this result provides a stability estimate for Sobolev norms of arbitrary degree s.

4. SCATTERED HERMITE INTERPOLATION ON THE REAL LINE

Estimates (3.4) and (3.5) can be used to provide another interesting result on interpolation on the real line. Since B acts as a bounded and positive definite operator on the sequence space $\ell_2(\mathbf{Z})$, we can find, for any data vector $d \in \ell_2(\mathbf{Z})$, a unique vector $a \in \ell_2(\mathbf{Z})$ such that $Ba = d$. In other words (with e_i, $i \in \mathbf{Z}$, denoting the canonical unit vectors in $\ell_2(\mathbf{Z})$)

$$\langle Ba, e_i \rangle = \left\langle h * \mu_i, \sum_{j \in \mathbf{Z}} \overline{a}_j h * \mu_j \right\rangle = d_i, \quad i \in \mathbf{Z}, \tag{4.1}$$

and

$$\gamma\, \|a\|_{\ell_2(\mathbf{Z})} \leqslant \|d\|_{\ell_2(\mathbf{Z})} \leqslant \Gamma\, \|a\|_{\ell_2(\mathbf{Z})} \tag{4.2}$$

for the same constants $0 < \gamma \leqslant \Gamma < \infty$ as in Theorem 3.2.

Now, for any $a = (a_j)_{j \in \mathbf{Z}} \in \ell_2(\mathbf{Z})$, using (3.7),

$$\begin{aligned} \left\langle h * \mu_i, \sum_{j \in \mathbf{Z}} \overline{a}_j h * \mu_j \right\rangle &= \frac{1}{2\pi} \left\langle h^\wedge \mu_i^\wedge, \sum_{j \in \mathbf{Z}} \overline{a}_j h^\wedge \mu_j^\wedge \right\rangle \\ &= \frac{\sqrt{d_{i,m}}}{2\pi} \left\langle B_{i,m}^\wedge, G \sum_{j \in \mathbf{Z}} \overline{a}_j \sqrt{d_{j,m}}\, B_{j,m}^\wedge \right\rangle \\ &= \frac{\sqrt{d_{i,m}}}{2\pi} \langle B_{i,m}^\wedge, G\, \sigma_{\bar{a},m}^\wedge \rangle, \end{aligned}$$

and from (1.5)

$$\mu_i(f) = \frac{\sqrt{d_{i,m}}}{2\pi} \langle B_{i,m}^\wedge, (f^{(m)})^\wedge \rangle.$$

Comparing these two identities we find the following.

THEOREM 4.1. *Suppose that* (3.1) *and Assumption* 1 *are satisfied. Then for any data vector* $d \in \ell_2(\mathbf{Z})$, *there is a unique spline*

$$\sigma = \sigma_{\bar{a},m} = \sum_{j \in \mathbf{Z}} \bar{a}_j \sqrt{d_{j,m}}\, B_{j,m} \in S_{m,X}$$

such that

$$\mu_i(f) = d_i, \qquad i \in \mathbf{Z}, \tag{4.3}$$

for some mth *primitive* f *of* g *with* $(\bar{g})^\wedge = G\,\sigma_{\bar{a},m}^\wedge$. *Furthermore, we have the inequalities* (4.2).

Specializing to the case of polynomial splines, where $h(x) = x_+^{m-1}/(m-1)!$ and $G(\xi) \equiv 1$, this gives

COROLLARY 4.2. *Suppose that* (3.1) *and Assumption* 1 *are satisfied. Then, for any data vector* $d \in \ell_2(\mathbf{Z})$ *there is a unique spline* $\sigma_{a,m} = \sum_{j \in \mathbf{Z}} a_j \sqrt{d_{j,m}}\, B_{j,m} \in S_{m,X}$ *such that* $\mu_i(f_a) = d_i$, $i \in \mathbf{Z}$, *for some* mth *primitive* f_a *of* $\sigma_{a,m}$.

In this corollary, we can take any mth primitive, since the divided difference annihilates polynomials of degree less than m. Since f_a is a spline of order $2m$, we can find a sequence $b = (b_i)_{i \in \mathbf{Z}}$ such that

$$f_a = \sigma_{b,2m} = \sum_{i \in \mathbf{Z}} b_i \sqrt{d_{i,2m}}\, B_{i,2m}.$$

The differentiation formula for a B-spline series in Lemma 1.2 now shows that necessarily

$$a = Db \qquad \text{with} \quad D := D_{m+1} D_{m+2} \cdots D_{2m}$$

which is a bi-infinite, lower m-banded matrix D. We point to the fact that our assumption $q_m > 0$ implies that $q_k > 0$ for $k = m+1, ..., 2m$ as well, showing at the same time that the entries of D are uniformly bounded.

The above results describe interpolation of the linear functionals $\mu_i(f)$, $i \in \mathbf{Z}$. With more restrictive assumptions on the knot set X, however, it is possible to obtain results for Lagrange- and Hermite-interpolation of data vectors in $\ell_2(\mathbf{Z})$.

For this purpose, we introduce the spaces

$$\tilde{V}_{2m,X} := \{f;\, f^{(m)} \in L_2(\mathbf{R}) \text{ and } (\overline{f^{(m)}})^\wedge = G\sigma^\wedge \text{ for some } \sigma \in S_{m,X}\} \tag{4.4}$$

and

$$V_{2m,X} := \tilde{V}_{2m,X} \cap L_2(\mathbf{R}). \tag{4.5}$$

In the spline case (i.e., if $G \equiv 1$) we obtain $V_{2m,X} = S_{2m,X}$. Note, that $V_{2m,X} \subset H^m(\mathbf{R})$ holds due to the standard estimate (see [6, Chap. 2, Theorem 5.6])

$$\|f^{(k)}\|_{L_2(\mathbf{R})} \leqslant \text{const}(\|f\|_{L_2(\mathbf{R})} + \|f^{(m)}\|_{L_2(\mathbf{R})}), \qquad 1 \leqslant k \leqslant m-1, \quad f \in H^m(\mathbf{R}). \tag{4.6}$$

We first consider the simple knot case, i.e., the case where the knot sequence has order 1. Here, we can apply (1.11) several times to see that

$$\mu^{(m)} = (-1)^{m-1} D_m^* D_{m-1}^* \cdots D_2^* \mu^{(1)} \tag{4.7}$$

where

$$\mu^{(1)} = (\mu_{i,1})_{i \in \mathbf{Z}} = (d_{i,1}^{-1/2}(v_{i+1,0} - v_{i,0}))_{i \in \mathbf{Z}}. \tag{4.8}$$

Thus using the definition of D_m as in Lemma 1.2 for $m=1$ with $d_{j,0} := 1$, we put

$$C := (-1)^m D_m^* D_{m-1}^* \cdots D_1^*. \tag{4.9}$$

This defines a bi-infinite upper m-banded matrix. It has been used as a preconditioner for unbounded operators in previous work of the authors and co-workers [5, 10].

Theorem 4.1 now takes the following form.

THEOREM 4.3. *Assume that the knot sequence X of order* 1 *satisfies* (3.1), *and let Assumption* 1 *be satisfied. Then for any bi-infinite data vector c such that $d := C c \in \ell_2(\mathbf{Z})$, there is a unique function $f = f_c \in \tilde{V}_{2m,X}$ such that*

$$f_c(x_i) = c_i, \qquad i \in \mathbf{Z}.$$

Proof. The existence of an interpolating function $f \in \tilde{V}_{2m,X}$ is verified by taking the mth primitive f in Theorem 4.1 to satisfy $f(x_i) = c_i$, $i = 1, ..., m$. Then, by (4.3) and (4.9), f necessarily interpolates all data c_i, $i \in \mathbf{Z}$. The uniqueness follows from considering the homogeneous equations, i.e., assuming that $c = d = 0$. Then $\sigma_{\bar{a},m} = 0$, and hence f is a polynomial of degree less than m. From $f(x_i) = 0$, $i = 1, ..., m$, we find that $f = 0$. ∎

It should be noted that the interpolant f_c is the unique element from

$$\mathscr{F}_c := \left\{ f;\ \frac{(\overline{f^{(m)}})^\wedge}{G} \in L_2(\mathbf{R}),\ f(x_i) = c_i,\ i \in \mathbf{Z} \right\}$$

minimizing the weighted seminorm

$$|f|_G := \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|(f^{(m)})^{\wedge}(\xi)|^2}{G^2(\xi)} \, d\xi \right)^{1/2}. \tag{4.10}$$

This can be seen as in [6, Chap. 5.6]. The main point to observe here is the closedness of the space of functions $\{f^{(m)}; f \in \tilde{V}_{2m, X}\}$ in the topology of the weighted seminorm (4.10).

It is also clear that Theorem 4.3 applies to data $c \in \ell_2(\mathbf{Z})$ in case the knots of X are minimally separated of order 1, i.e., $q_1 > 0$. (Note that for the proof of Theorem 4.3 we have only assumed that $q_m > 0$.) In this case, C is a banded matrix with uniformly bounded entries. It therefore defines a bounded operator on $\ell_2(\mathbf{Z})$, and Theorem 4.3 is applicable for arbitrary data sequences. This proves the existence part in the following general interpolation theorem.

THEOREM 4.4. *Suppose that the knot sequence X is minimally separated of order* 1 *and Assumption* 1 *is satisfied.*

(a) *For any data vector $c \in \ell_2(\mathbf{Z})$, there is a unique function $f \in \tilde{V}_{2m, X}$ with $f(x_i) = c_i$, $i \in \mathbf{Z}$.*

(b) *If X has bounded global mesh ratio*

$$\sup_{j,k \in \mathbf{Z}} \frac{x_{j+1} - x_j}{x_{k+1} - x_k} < \infty, \tag{4.11}$$

then we have that $f \in H^m(\mathbf{R})$, and there exist constants $0 < \gamma_1 \leqslant \gamma_2 < \infty$ (depending only on X and m) such that

$$\gamma_1 \|c\|_{\ell_2(\mathbf{Z})} \leqslant \|f\|_{H^m(\mathbf{R})} \leqslant \gamma_2 \|c\|_{\ell_2(\mathbf{Z})}. \tag{4.12}$$

For the proof we only need to show that inequalities (4.12) hold. The main tools are the Sobolev imbedding theorem and the Bramble–Hilbert Lemma, see [3]. We state this as a separate result.

LEMMA 4.5. *Let X be a bi-infinite knot sequence with bounded global mesh ratio* (4.11), *and let $m \in \mathbf{N}$. Then the following norm*

$$\|f\|_{m, X} := (\|f|_X\|^2_{\ell_2(\mathbf{Z})} + \|f^{(m)}\|^2_{L_2(\mathbf{R})})^{1/2}, \qquad f \in H^m(\mathbf{R}), \tag{4.13}$$

is equivalent to the Sobolev norm $\|f\|_{m,2} := (\sum_{k=0}^{m} \|f^{(k)}\|^2_{L_2(\mathbf{R})})^{1/2}$ (which is known to be equivalent to the norm $\|f\|_{H^m(\mathbf{R})}$ as defined in Section 2).

Proof. Since X is minimally separated of order 1 by (4.11), we have $q_1 > 0$. Hence, by the Sobolev imbedding theorem (cf. [1, Chap. V, Theorem 5.4]), there is a constant γ_1 which only depends on q_1 such that

$$|f(x_i)| \leqslant \gamma_1 \|f\|_{m,\,2,\,I_i}, \qquad f \in H^m(I_i),$$

where $I_i := (x_i - q_1/2,\ x_i + q_1/2)$, $i \in \mathbf{Z}$, are disjoint intervals. Summation over $i \in \mathbf{Z}$ gives

$$\|f|_X\|^2_{\ell_2(\mathbf{Z})} \leqslant \gamma_1^2\, \|f\|^2_{m,\,2},$$

and this shows that $\|f\|_{m,\,X} \leqslant \text{const}\,\|f\|_{m,\,2}$ holds for all $f \in H^m(\mathbf{R})$.

On the other hand, (4.11) also yields a parameter $\alpha > 0$ such that any open interval of length 2α contains at least m points from X. For fixed $x \in \mathbf{R}$, we let $R_x := (x-\alpha,\ x+\alpha)$ and choose m points $x_{k_i} \in R_x \cap X$, $1 \leqslant i \leqslant m$. By the imbedding theorem, the point evaluation functionals $\ell_i(f) := f(x_{k_i})$ are bounded by the Sobolev norm on R_x,

$$|\ell_i(f)| \leqslant \gamma_2\, \|f\|_{m,\,2,\,R_x},$$

where γ_2 depends only on α. Furthermore, since $q_1 > 0$, the corresponding Lagrange polynomials p_i, $1 \leqslant i \leqslant m$, of degree less than m satisfy

$$\ell_i(p_j) = \delta_{ij} \qquad \text{and} \qquad \sum_{i=1}^{m} |p_i(x)|^2 \leqslant \gamma_3,$$

where γ_3 only depends on α, q_1, and m. Hence, the linear functional

$$F_x\colon f \mapsto f(x) - \sum_{i=1}^{m} \ell_i(f)\, p_i(x), \qquad f \in H^m(R_x),$$

is again bounded. It obviously annihilates all polynomials of degree less than m. As a consequence of the Bramble–Hilbert Lemma [3, Theorem 2], we obtain that

$$|F_x(f)| \leqslant \gamma_4\, \|f^{(m)}\|_{L_2(R_x)}, \qquad f \in H^m(R_x),$$

with γ_4 only depending on α, q_1, and m.

From this inequality we immediately find

$$\begin{aligned} |f(x)|^2 &= |F_x(f) + \sum_{i=1}^{m} \ell_i(f)\, p_i(x)|^2 \leqslant 2\, |F_x(f)|^2 + 2\gamma_3 \sum_{i=1}^{m} |\ell_i(f)|^2 \\ &\leqslant 2\gamma_4^2 \int_{R_x} |f^{(m)}(t)|^2\, dt + 2\gamma_3 \sum_{x_k \in R_x \cap X} |f(x_k)|^2 \\ &\leqslant \text{const} \left\{ \int_{R_x} |f^{(m)}(t)|^2\, dt + \int_{R_x} |f(t)|^2\, d\sigma_X(t) \right\} \qquad (4.14) \end{aligned}$$

with $\sigma_X = \sum_{k \in \mathbf{Z}} \delta_{x_k}$. Upon integration we obtain

$$\int_{\mathbf{R}} |f(x)|^2 \, dx \leqslant \text{const} \left\{ \int_{\mathbf{R}} |f^{(m)}(t)|^2 \int_{t-\alpha}^{t+\alpha} dx \, dt + \int_{\mathbf{R}} |f(t)|^2 \int_{t-\alpha}^{t+\alpha} dx \, d\sigma_X(t) \right\}$$
$$= 2\alpha \, \text{const} \left\{ \int_{\mathbf{R}} |f^{(m)}(t)|^2 \, dt + \sum_{k \in \mathbf{Z}} |f(x_k)|^2 \right\}.$$

Thus we have shown that $\|f\|_{L_2(\mathbf{R})} \leqslant \text{const} \, \|f\|_{m,X}$ holds for all $f \in H^m(\mathbf{R})$, with the constant depending only on m, q_1, and the mesh ratio (4.11). Bounding the intermediate derivatives of f by $\|f\|_{m,X}$ is done using the standard estimate (4.6). Therefore, we have proved the equivalence of the norm (4.13) and the Sobolev norm. ∎

We have the feeling that the result of Lemma 4.5 is not original. However, we could not find it in the standard literature on Sobolev spaces, and for this reason we wanted to include a proof.

Proof of Theorem 4.4. Let f be the unique interpolant from $\tilde{V}_{2m,X}$ to given data $c \in \ell_2(\mathbf{Z})$. Then, with the same notations as in Theorem 4.1 and $d = Cc$, we have

$$\|f^{(m)}\|^2_{L_2(\mathbf{R})} = \frac{1}{2\pi} \int_{\mathbf{R}} G^2(\xi) \, |\sigma_{\hat{a},m}(\xi)|^2 \, d\xi \leqslant \Gamma^2 \, \|a\|^2_{\ell_2(\mathbf{Z})} \leqslant \frac{\Gamma^2}{\gamma^2} \, \|Cc\|^2_{\ell_2(\mathbf{Z})},$$

the latter according to (4.2). Since C is a bounded operator by (4.11), we obtain for the equivalent norm (4.13)

$$\|f\|^2_{m,X} \leqslant \left(1 + \left(\frac{\Gamma \, \|C\|}{\gamma}\right)^2\right) \|c\|^2_{\ell_2(\mathbf{Z})},$$

which gives the right-hand side of (4.12). In particular, f is thus shown to be an element of $H^m(\mathbf{R})$. But then, the left-hand side of (4.12) is just another consequence of Lemma 4.5. ∎

It is in order to add two remarks to Theorem 4.4. The stability result holds true for the modified norm

$$\|f\|_G := (\|f|_X\|^2_{\ell_2(\mathbf{Z})} + |f|^2_G)^{1/2}$$

instead of the Sobolev norm, with $|\cdot|_G$ as in (4.10) and different constants γ_1, γ_2 in (4.12). A related result for multivariate gridded data was already elaborated in [9, Section 4], where the close relation of cardinal interpolation with L_2-projection was shown. There is also some connection to the work by Madych and Nelson [11]; but they follow a different approach to the interpolation problem.

Specializing to the case of spline functions we have shown the following result.

COROLLARY 4.6. *Suppose that the assumptions of Theorem* 4.4 *are satisfied, and let X have bounded global mesh ratio. Then, for any data vector* $c \in \ell_2(\mathbf{Z})$, *there is a unique spline function* $\sigma_{b,2m} = \sum_{j \in \mathbf{Z}} b_j \sqrt{d_{j,2m}}\, B_{j,2m} \in S_{2m,X}$ *with*

$$\sigma_{b,2m}(x_i) = c_i\,, \qquad i \in \mathbf{Z},$$

and there exist constants $0 < \gamma_1 \leqslant \gamma_2 < \infty$ (*depending only on m and X*) *such that*

$$\gamma_1 \, \|c\|_{\ell_2(\mathbf{Z})} \leqslant \|\sigma_{b,2m}\|_{L_2(\mathbf{R})} \leqslant \gamma_2 \, \|c\|_{\ell_2(\mathbf{Z})}. \tag{4.15}$$

In this result, we can take the L_2-norm instead of the Sobolev norm in (4.15), since both norms are equivalent on the spline space $S_{2m,X}$ by Lemma 1.2. We thus have shown the stability of the spline interpolation operator for splines of even order with a bi-infinite knot sequence. Note that in contrast to the proof in [6, Chap. 13, Theorem 5.2] we have not appealed to Demko's result.

In the multiple knot case, we can argue as follows. Let X be an extended knot sequence of order m, and let $\Xi = \{\cdots < \xi_k < \xi_{k+1} < \cdots\}$ be the subsequence of *distinct* points from X. Let c be a bi-infinite data vector. Then $f|_X = c$ means that f interpolates the data c in the following way. If x_i is a simple knot from X, then $f(x_i) = c_i$ as before. However, if $\operatorname{ord}(x_i) > 1$, for example $x_i = x_j$ with

$$x_{j-1} < x_j = x_{j+1} = \cdots = x_{j+p} < x_{j+p+1},$$

then the interpolation conditions at x_i become

$$\nu_{j,k}(f) = \frac{1}{k!} f^{(k)}(x_j) = c_{j+k}, \qquad k = 0, ..., p.$$

Given an extended knot sequence of order m, we thus see that at most $(m-1)$st derivatives will be involved.

It is now important to notify that by putting

$$\nu(f) := f|_X$$

we can write

$$\mu^{(m)}(f) = C\nu(f) \tag{4.16}$$

with a banded matrix C which is a modification of the matrix (4.9). This matrix has uniformly bounded entries provided that the sequence Ξ is minimally separated of order 1. This can easily be seen from the recursive definition (1.2), since division by differences of distinct knots is not critical in this case. Depending on the requirements posed on the knot set Ξ we find the following results.

THEOREM 4.7. *Let X be an extended knot sequence of order m such that* (3.1) *holds, and suppose that Assumption* 1 *is satisfied.*

(a) *For any data vector $c \in \ell_2(\mathbf{Z})$ with $d := Cc \in \ell_2(\mathbf{Z})$, there is a unique function $f \in \tilde{V}_{2m,X}$ with $f|_X = c$.*

(b) *If the reduced knot set Ξ is minimally separated of order* 1, *then* (a) *holds for any data vector $c \in \ell_2(\mathbf{Z})$.*

(c) *If the reduced knot set Ξ has bounded global mesh ratio*

$$\sup_{j,k \in \mathbf{Z}} \frac{\xi_{j+1} - \xi_j}{\xi_{k+1} - \xi_k} < \infty,$$

then $f \in H^m(\mathbf{R})$, and there exist constants $0 < \gamma_1 \leqslant \gamma_2 < \infty$ (depending only on X and m) such that

$$\gamma_1 \|c\|_{\ell_2(\mathbf{Z})} \leqslant \|f\|_{H^m(\mathbf{R})} \leqslant \gamma_2 \|c\|_{\ell_2(\mathbf{Z})}. \tag{4.17}$$

Proof. The proof is very much the same as before. The only difference in part (a) is to choose m knots $x_j, ..., x_{j+m-1}$ with $x_{j-1} < x_j$ in order to obtain a regular Hermite interpolation problem. Part (b) follows from (a) and the boundedness of the operator C. For part (c) we have to give a few more details. The essential part is to show that an analogue of Lemma 4.5 holds true for the modified norm (4.13), where the correct interpretation of $f|_X$ is used. The Sobolev imbedding theorem can be applied in order to find a uniform bound for all functionals in $v(f)$, since only derivatives of order less than m are involved. This gives the first part of the proof of the lemma. The second part is proved in the same way as before by choosing m points in $R_x \cap \Xi$ and using Lagrange interpolation at these points. The only change is in (4.14) and the subsequent formula, where the expression $\sum_{x_k \in R_x \cap X} |f(x_k)|^2$ is substituted by the norm of the vector

$$\|f|_{(R_x \cap X)}\|_{\ell_2}^2$$

with an appropriate interpretation of multiple knots by function and derivative values. No other changes are needed for the proof of part (c). ∎

In order to explain this result in more detail, we consider the special case of

$$\operatorname{ord}(x_i) = 2\,, \qquad i \in \mathbf{Z},$$

i.e., the case of Hermite interpolation with double knots. In this case,

$$X = \{ \cdots < x_{2i} = x_{2i+1} < x_{2i+2} = x_{2i+3} \cdots \}$$

and we let

$$\xi_i := x_{2i}, \qquad i \in \mathbf{Z}\,.$$

The data vector $v(f)$ now takes the form

$$v(f) = (\ldots, f(\xi_i), f'(\xi_i), f(\xi_{i+1}), f'(\xi_{i+1}), \ldots)$$

where the index 0 corresponds to $f(\xi_0)$, the index 1 to $f'(\xi_0)$, etc. The second divided differences $\mu_{i,2}$, $i \in \mathbf{Z}$, are given by the vector

$$\mu^{(2)}(f) = Ev(f), \qquad \text{where} \quad E = (e_{ij})_{i,j \in \mathbf{Z}},$$

and the entries of E are obtained from

$$\mu_{2i,2}(f) = -\frac{\sqrt{2}}{(\xi_{i+1} - \xi_i)^{3/2}} (f(\xi_i) + (\xi_{i+1} - \xi_i) f'(\xi_i) - f(\xi_{i+1})),$$

$$\mu_{2i+1,2}(f) = +\frac{\sqrt{2}}{(\xi_{i+1} - \xi_i)^{3/2}} (f(\xi_i) - f(\xi_{i+1}) + (\xi_{i+1} - \xi_i) f'(\xi_{i+1})), \quad i \in \mathbf{Z}.$$

Due to the Hermite data, E is no longer triangular, but still 2-banded. Each row has 3 non-zero entries. If $f(x_0)$ and $f'(x_0)$ are specified (as in the proof of Theorem 4.3), then $f|_X$ can be recursively determined from the values $\mu^{(2)}(f)$. Higher order differences are obtained as in (4.7),

$$\mu^{(m)}(f) = (-1)^{m-2} D_m^* D_{m-1}^* \cdots D_3^* Ev(f) =: Cv(f).$$

It is clear that the entries of E are uniformly bounded, provided that the knots $\{\xi_i; i \in \mathbf{Z}\}$ are minimally separated of order 1. The same is true for the matrix C in case $m \geqslant 3$. Part (c) of Theorem 4.7 implies that the Hermite spline interpolation problem with data $c \in \ell_2(\mathbf{Z})$ is always solvable, and that the stability estimate

$$\gamma_1 \|c\|_{\ell_2(\mathbf{Z})} \leqslant \|\sigma_{b,\,2m}\|_{L_2(\mathbf{R})} \leqslant \gamma_2 \|c\|_{\ell_2(\mathbf{Z})}$$

holds true for the unique Hermite spline interpolant $\sigma_{b,2m} \in S_{2m,X}$, provided that the set Ξ has bounded global mesh ratio.

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